

Bounds for the Second Hankel Determinant of Certain Univalent Functions

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ABSTRACT. The estimates for the second Hankel determinant $a_2a_4 - a_3^2$ of analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ for which either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to certain analytic function are investigated. The estimates for the Hankel determinant for two other classes are also obtained. In particular, the estimates for the Hankel determinant of strongly starlike, parabolic starlike, lemniscate starlike functions are obtained.

1. Introduction

Let \mathcal{A} denote the class of all analytic functions

$$(1.1) \quad f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The *Hankel determinants* $H_q(n)$, ($n = 1, 2, \dots, q = 1, 2, \dots$) of the function f are defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

Hankel determinants are useful, for example, in showing that a function of bounded characteristic in \mathbb{D} , i.e., a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [7]. For the use of Hankel determinant in the study of meromorphic functions, see [40], and various properties of these determinants can be found in [38, Chapter 4]. In 1966, Pommerenke [32] investigated the Hankel determinant of areally mean p -valent functions, univalent functions as well as for starlike functions. In [33], he proved that the Hankel determinants of univalent functions satisfy

$$|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \dots, q = 2, 3, \dots),$$

where $\beta > 1/4000$ and K depends only on q . Later, Hayman [15] proved that $|H_2(n)| < An^{1/2}$, ($n = 1, 2, \dots$; A an absolute constant) for areally mean univalent functions. In [21–23], the estimates for Hankel determinant for areally mean p -valent functions were investigated. ElHosh obtained bounds for Hankel determinants of univalent functions with positive Hayman index α [9] and of k -fold symmetric and close-to-convex functions [10]. For bounds on the Hankel determinants of close-to-convex functions, see [24–26]. Noor studied the Hankel determinant of Bazilevic functions in [27] and of functions with bounded boundary rotation in [28–31]. In the recent years, several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent functions [5, 12–14, 16, 18–20]. The Hankel determinant $H_2(1) = a_3 - a_2^2$ is the well known Fekete-Szegő functional. For results

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related to this functional, see [2, 4]. The second Hankel determinant $H_2(2)$ is given by $H_2(2) = a_2a_4 - a_3^2$.

An analytic function f is *subordinate* to an analytic function g , written $f(z) \prec g(z)$, if there is an analytic function $w : \mathbb{D} \rightarrow \mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. Ma and Minda [17] unified various subclasses of starlike (\mathcal{S}^*) and convex functions (\mathcal{C}) by requiring that either of the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a function ϕ with positive real part in the unit disk \mathbb{D} , $\phi(0) = 1$, $\phi'(0) > 0$, ϕ maps \mathbb{D} onto a region starlike with respect to 1 and symmetric with respect to the real axis. He obtained distortion, growth and covering estimates as well as bounds for the initial coefficients of the unified classes.

The bounds for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ are obtained for functions belonging to these subclasses of Ma-Minda starlike and convex functions in Section 2. In section 3, the problem is investigated for two other related classes defined by subordination. In proving our results, we do not assume the univalence or starlikeness of ϕ as they were required only in obtaining the distortion, growth estimates and the convolution theorems. The classes introduced by subordination naturally include several well known classes of univalent functions and the results for some of these special classes are indicated as corollaries.

Let \mathcal{P} be the class of *functions with positive real part* consisting of all analytic functions $p : \mathbb{D} \rightarrow \mathbb{C}$ satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$. We need the following results about the functions belonging to the class \mathcal{P} :

LEMMA 1.1. [8] *If the function $p \in \mathcal{P}$ is given by the series*

$$(1.2) \quad p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots,$$

then the following sharp estimate holds:

$$(1.3) \quad |c_n| \leq 2 \quad (n = 1, 2, \dots).$$

LEMMA 1.2. [11] *If the function $p \in \mathcal{P}$ is given by the series (1.2), then*

$$(1.4) \quad 2c_2 = c_1^2 + x(4 - c_1^2),$$

$$(1.5) \quad 4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2. Second Hankel determinant of Ma-Minda starlike/convex functions

Various subclasses of starlike functions are characterized by the quantity $zf'(z)/f(z)$ lying in some domain in the right half-plane. For example, f is strongly starlike of order β if $zf'(z)/f(z)$ lies in a sector $|\arg w| < \beta\pi/2$ while it is starlike of order α if $zf'(z)/f(z)$ lies in the half-plane $\operatorname{Re} w > \alpha$. The various subclasses of starlike functions were unified by subordination in [17]. The following definition of the class of Ma-Minda starlike functions is the same as the one in [17] except for the omission of starlikeness assumption of ϕ .

DEFINITION 2.1. Let $\phi : \mathbb{D} \rightarrow \mathbb{C}$ be analytic and the Maclaurin series of ϕ is given by

$$(2.1) \quad \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots, \quad (B_1, B_2 \in \mathbb{R}, B_1 > 0).$$

The class $\mathcal{S}^*(\varphi)$ of *Ma-Minda starlike functions with respect to φ* consists of functions $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

For the function φ given by $\varphi_\alpha(z) := (1 + (1 - 2\alpha)z)/(1 - z)$, $0 < \alpha \leq 1$, the class $\mathcal{S}^*(\alpha) := \mathcal{S}^*(\varphi_\alpha)$ is the well-known class of starlike functions of order α . Let

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2} \left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2.$$

Then the class

$$\mathcal{S}_P^* := \mathcal{S}^*(\varphi_{PAR}) = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\}$$

is the *parabolic starlike functions* introduced by Rønning [34]. For a survey of parabolic starlike functions and the related class of uniformly convex functions, see [3]. For $0 < \beta \leq 1$, the class

$$\mathcal{S}_\beta^* := \mathcal{S}^* \left(\left(\frac{1+z}{1-z} \right)^\beta \right) = \left\{ f \in \mathcal{A} : \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\beta\pi}{2} \right\}$$

is the familiar class of *strongly starlike functions of order β* . The class

$$\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z}) = \left\{ f \in \mathcal{A} : \left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \right\}$$

is the class of *lemniscate starlike functions* studied in [37].

THEOREM 2.1. *Let the function $f \in \mathcal{S}^*(\varphi)$ be given by (1.1).*

(1) *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \leq B_1, \quad 4B_1^4 - 16B_1|B_3| + 12B_2^2 - 6B_1|B_2| + 9B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{4}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions*

$$|B_2| \geq B_1, \quad 4B_1^4 - 16B_1|B_3| + 12B_2^2 - 2B_1|B_2| + 5B_1^2 \leq 0,$$

or the conditions

$$|B_2| \leq B_1, \quad 4B_1^4 - 16B_1|B_3| + 12B_2^2 - 6B_1|B_2| + 9B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{48} (-4B_1^4 + 16B_1|B_3| - 12B_2^2 + 6B_1|B_2| + 3B_1^2).$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$|B_2| > B_1, \quad 4B_1^4 - 16B_1|B_3| + 12B_2^2 - 2B_1|B_2| + 5B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{12} \left(\frac{12B_1^4 - 48B_1|B_3| + 40B_2^2 - 2B_1|B_2| + 7B_1^2}{4B_1^4 - 16B_1|B_3| + 12B_2^2 + 2B_1|B_2| + B_1^2} \right).$$

PROOF. Since $f \in \mathcal{S}^*(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in \mathbb{D} such that

$$(2.2) \quad \frac{zf'(z)}{f(z)} = \varphi(w(z)).$$

Define the functions p_1 by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \dots$$

or equivalently,

$$(2.3) \quad w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left(c_1z + \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \dots \right).$$

Then p_1 is analytic in \mathbb{D} with $p_1(0) = 1$ and has positive real part in \mathbb{D} . By using (2.3) together with (2.1), it is evident that

$$(2.4) \quad \varphi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2 \right) z^2 + \dots.$$

Since

$$(2.5) \quad \frac{zf'(z)}{f(z)} = 1 + a_2z + (-a_2^2 + 2a_3)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \dots,$$

it follows by (2.2), (2.4) and (2.5) that

$$a_2 = \frac{B_1c_1}{2},$$

$$a_3 = \frac{1}{8} [(B_1^2 - B_1 + B_2)c_1^2 + 2B_1c_2],$$

$$a_4 = \frac{1}{48} [(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_3)c_1^3 + 2(3B_1^2 - 4B_1 + 4B_2)c_1c_2 + 8B_1c_3].$$

Therefore

$$a_2a_4 - a_3^2 = \frac{B_1}{96} \left[c_1^4 \left(-\frac{B_1^3}{2} + \frac{B_1}{2} - B_2 + 2B_3 - \frac{3B_2^2}{2B_1} \right) + 2c_2c_1^2(B_2 - B_1) + 8B_1c_1c_3 - 6B_1c_2^2 \right].$$

Let

$$d_1 = 8B_1, \quad d_2 = 2(B_2 - B_1),$$

$$(2.6) \quad d_3 = -6B_1, \quad d_4 = -\frac{B_1^3}{2} + \frac{B_1}{2} - B_2 + 2B_3 - \frac{3B_2^2}{2B_1},$$

$$T = \frac{B_1}{96}.$$

Then

$$(2.7) \quad |a_2a_4 - a_3^2| = T|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|.$$

Since the function $p(e^{i\theta}z)$ ($\theta \in \mathbb{R}$) is in the class \mathcal{P} for any $p \in \mathcal{P}$, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c$, $c \in [0, 2]$. Substituting the values of c_2 and c_3 respectively from (1.4) and (1.5) in (2.7), it follows that

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{T}{4} \left| c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3) \right. \\ &\quad \left. + (4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z \right|. \end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (2.6), yield

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{T}{4} \left[c^4 \left(-2B_1^3 + 8|B_3| - 6\frac{B_2^2}{B_1} \right) + 4|B_2|\mu c^2(4 - c^2) \right. \\ &\quad \left. + \mu^2(4 - c^2)(2B_1c^2 + 24B_1) + 16B_1c(4 - c^2)(1 - \mu^2) \right] \\ (2.8) \quad &= T \left[\frac{c^4}{4} \left(-2B_1^3 + 8|B_3| - 6\frac{B_2^2}{B_1} \right) + 4B_1c(4 - c^2) + |B_2|(4 - c^2)\mu c^2 \right. \\ &\quad \left. + \frac{B_1}{2}\mu^2(4 - c^2)(c - 6)(c - 2) \right] \\ &\equiv F(c, \mu). \end{aligned}$$

Note that for $(c, \mu) \in [0, 2] \times [0, 1]$, differentiating $F(c, \mu)$ in (2.8) partially with respect to μ yields

$$(2.9) \quad \frac{\partial F}{\partial \mu} = T [|B_2|(4 - c^2) + B_1\mu(4 - c^2)(c - 2)(c - 6)].$$

Then for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$, it is clear from (2.9) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c, \mu)$ is an increasing function of μ . Hence for fixed $c \in [0, 2]$, the maximum of $F(c, \mu)$ occurs at $\mu = 1$, and

$$\max F(c, \mu) = F(c, 1) \equiv G(c).$$

Also note that

$$G(c) = \frac{B_1}{96} \left[\frac{c^4}{4} \left(-2B_1^3 + 8|B_3| - 6\frac{B_2^2}{B_1} - |B_2| - \frac{B_1}{2} \right) + 4c^2(|B_2| - B_1) + 24B_1 \right].$$

Let

$$\begin{aligned} (2.10) \quad P &= \frac{1}{4} \left(-2B_1^3 + 8|B_3| - 6\frac{B_2^2}{B_1} - |B_2| - \frac{B_1}{2} \right), \\ Q &= 4(|B_2| - B_1), \\ R &= 24B_1. \end{aligned}$$

Since

$$(2.11) \quad \max_{0 \leq t \leq 4} (Pt^2 + Qt + R) = \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8}, \end{cases}$$

we have

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{96} \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given by (2.10). □

REMARK 2.1. When $B_1 = B_2 = B_3 = 2$, Theorem 2.1 reduces to [16, Theorem 3.1].

COROLLARY 2.1.

- (1) If $f \in \mathcal{S}^*(\alpha)$, for $0 < \alpha \leq 3/4$, $|a_2a_4 - a_3^2| \leq (1 - \alpha)^2$. And for $3/4 \leq \alpha \leq 1$, $|a_2a_4 - a_3^2| \leq (1 - \alpha)^2[13 - 16(1 - \alpha)^2]/12$.
- (2) If $f \in \mathcal{S}_L^*$, then $|a_2a_4 - a_3^2| \leq 1/16 = 0.0625$.
- (3) If $f \in \mathcal{S}_P^*$, then $|a_2a_4 - a_3^2| \leq 16/\pi^4 \approx 0.164255$.
- (4) If $f \in \mathcal{S}_\beta^*$, then $|a_2a_4 - a_3^2| \leq \beta^2$.

DEFINITION 2.2. Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be analytic and $\varphi(z)$ is given as in (2.1). The class $\mathcal{C}(\varphi)$ of Ma-Minda convex functions with respect to φ consists of functions f satisfying the subordination

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z).$$

THEOREM 2.2. Let the function $f \in \mathcal{C}(\varphi)$ be given by (1.1).

- (1) If B_1, B_2 and B_3 satisfy the conditions

$$B_1^2 + 4|B_2| - 2B_1 \leq 0, \quad B_1^4 - B_1^2|B_2| - 6B_1|B_3| + 4B_2^2 + 4B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{36}.$$

- (2) If B_1, B_2 and B_3 satisfy the conditions

$$B_1^2 + 4|B_2| - 2B_1 \geq 0, \quad 2B_1^4 - 2B_1^2|B_2| - 12B_1|B_3| + 8B_2^2 + 4B_1|B_2| + B_1^3 + 6B_1^2 \leq 0,$$

or the conditions

$$B_1^2 + 4|B_2| - 2B_1 \leq 0, \quad B_1^4 - B_1^2|B_2| - 6B_1|B_3| + 4B_2^2 + 4B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{144}(-B_1^4 + B_1^2|B_2| + 6B_1|B_3| - 4B_2^2).$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$B_1^2 + 4|B_2| - 2B_1 > 0, \quad 2B_1^4 - 2B_1^2|B_2| - 12B_1|B_3| + 8B_2^2 + 4B_1|B_2| + B_1^3 + 6B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{576} \left(\frac{17B_1^4 - 8B_1^2|B_2| - 96B_1|B_3| + 80B_2^2 + 12B_1^3 + 48B_1|B_2| + 36B_1^2}{B_1^4 - B_1^2|B_2| - 6B_1|B_3| + 4B_2^2 + B_1^3 + 4B_1|B_2| + 2B_1^2} \right).$$

PROOF. Since $f \in \mathcal{C}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in \mathbb{D} such that

$$(2.12) \quad 1 + \frac{zf''(z)}{f'(z)} = \varphi(w(z)).$$

Since

$$(2.13) \quad 1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2z + (-4a_2^2 + 6a_3)z^2 + (8a_2^3 - 18a_2a_3 + 12a_4)z^3 + \dots,$$

equations (2.4), (2.12) and (2.13) yield

$$\begin{aligned} a_2 &= \frac{B_1c_1}{4}, \\ a_3 &= \frac{1}{24} [(B_1^2 - B_1 + B_2)c_1^2 + 2B_1c_2], \\ a_4 &= \frac{1}{192} [(-4B_2 + 2B_1 + B_1^3 - 3B_1^2 + 3B_1B_2 + 2B_3)c_1^3 + 2(3B_1^2 - 4B_1 + 4B_2)c_1c_2 + 8B_1c_3]. \end{aligned}$$

Therefore

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{B_1}{768} \left[c_1^4 \left(-\frac{4}{3}B_2 + \frac{2}{3}B_1 - \frac{1}{3}B_1^3 - \frac{1}{3}B_1^2 + \frac{1}{3}B_1B_2 + 2B_3 - \frac{4}{3}\frac{B_2^2}{B_1} \right) \right. \\ &\quad \left. + \frac{2}{3}c_2c_1^2(B_1^2 - 4B_1 + 4B_2) + 8B_1c_1c_3 - \frac{16}{3}B_1c_2^2 \right]. \end{aligned}$$

By writing

$$\begin{aligned} d_1 &= 8B_1, \quad d_2 = \frac{2}{3}(B_1^2 - 4B_1 + 4B_2), \\ (2.14) \quad d_3 &= -\frac{16}{3}B_1, \quad d_4 = -\frac{4}{3}B_2 + \frac{2}{3}B_1 - \frac{1}{3}B_1^3 - \frac{1}{3}B_1^2 + \frac{1}{3}B_1B_2 + 2B_3 - \frac{4}{3}\frac{B_2^2}{B_1}, \\ T &= \frac{B_1}{768}, \end{aligned}$$

we have

$$(2.15) \quad |a_2a_4 - a_3^2| = T|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|.$$

Similar as in Theorems 2.1, it follows from (1.4) and (1.5) that

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{T}{4} |c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3) \\ &\quad + (4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z|. \end{aligned}$$

Replacing $|x|$ by μ and then substituting the values of d_1, d_2, d_3 and d_4 from (2.14) yield

$$\begin{aligned}
 |a_2 a_4 - a_3^2| &\leq \frac{T}{4} \left[c^4 \left(-\frac{4}{3} B_1^3 + \frac{4}{3} B_1 B_2 + 8 B_3 - \frac{16}{3} \frac{B_2^2}{B_1} \right) + 2\mu c^2 (4 - c^2) \left(\frac{2}{3} B_1^2 + \frac{8}{3} B_2 \right) \right. \\
 &\quad \left. + \mu^2 (4 - c^2) \left(\frac{8}{3} B_1 c^2 + \frac{64}{3} B_1 \right) + 16 B_1 c (4 - c^2) (1 - \mu^2) \right] \\
 (2.16) \quad &= T \left[\frac{c^4}{3} \left(-B_1^3 + B_1 |B_2| + 6 |B_3| - 4 \frac{B_2^2}{B_1} \right) + 4 B_1 c (4 - c^2) + \frac{1}{3} \mu c^2 (4 - c^2) (B_1^2 + 4 |B_2|) \right. \\
 &\quad \left. + \frac{2 B_1}{3} \mu^2 (4 - c^2) (c - 4) (c - 2) \right] \\
 &\equiv F(c, \mu).
 \end{aligned}$$

Again, differentiating $F(c, \mu)$ in (2.16) partially with respect to μ yield

$$(2.17) \quad \frac{\partial F}{\partial \mu} = T \left[\frac{c^2}{3} (4 - c^2) (B_1^2 + 4 |B_2|) + \frac{4 B_1}{3} \mu (4 - c^2) (c - 4) (c - 2) \right].$$

It is clear from (2.17) that $\frac{\partial F}{\partial \mu} > 0$. Thus $F(c, \mu)$ is an increasing function of μ for $0 < \mu < 1$ and for any fixed c with $0 < c < 2$. So the maximum of $F(c, \mu)$ occurs at $\mu = 1$ and

$$\max F(c, \mu) = F(c, 1) \equiv G(c).$$

Note that

$$\begin{aligned}
 G(c) &= T \left[\frac{c^4}{3} \left(-B_1^3 + B_1 |B_2| + 6 |B_3| - 4 \frac{B_2^2}{B_1} - B_1^2 - 4 |B_2| - 2 B_1 \right) \right. \\
 &\quad \left. + \frac{4}{3} c^2 (B_1^2 + 4 |B_2| - 2 B_1) + \frac{64}{3} B_1 \right].
 \end{aligned}$$

Let

$$\begin{aligned}
 P &= \frac{1}{3} \left(-B_1^3 + B_1 |B_2| + 6 |B_3| - 4 \frac{B_2^2}{B_1} - B_1^2 - 4 |B_2| - 2 B_1 \right), \\
 (2.18) \quad Q &= \frac{4}{3} (B_1^2 + 4 |B_2| - 2 B_1), \\
 R &= \frac{64}{3} B_1,
 \end{aligned}$$

By using (2.11), we have

$$|a_2 a_4 - a_3^2| \leq \frac{B_1}{768} \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given in (2.18). □

REMARK 2.2. For the choice of $\varphi(z) = (1+z)/(1-z)$, Theorem 2.2 reduces to [16, Theorem 3.2].

3. Further results on the second Hankel determinant

DEFINITION 3.1. Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be analytic and $\varphi(z)$ as given in (2.1). Let $0 \leq \gamma \leq 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}_\gamma^\tau(\varphi)$ if it satisfies the following subordination:

$$1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) \prec \varphi(z).$$

THEOREM 3.1. Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C} \setminus \{0\}$ and the function f as in (1.1) is in the class $\mathcal{R}_\gamma^\tau(\varphi)$. Also, let

$$p = \frac{8(1+\gamma)(1+3\gamma)}{9(1+2\gamma)^2}.$$

(1) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2|(1-p) + B_1(1-2p) \leq 0, \quad |B_1B_3 - pB_2^2| - pB_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2 B_1^2}{9(1+2\gamma)^2}.$$

(2) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2|(1-p) + B_1(1-2p) \geq 0, \quad 2|B_1B_3 - pB_2^2| - 2(1-p)B_1|B_2| - B_1 \geq 0,$$

or the conditions

$$2|B_2|(1-p) + B_1(1-2p) \leq 0, \quad |B_1B_3 - pB_2^2| - B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2}{8(1+\gamma)(1+3\gamma)} |B_3B_1 - pB_2^2|.$$

(3) If B_1, B_2 and B_3 satisfy the conditions

$$2|B_2|(1-p) + B_1(1-2p) > 0, \quad 2|B_1B_3 - pB_2^2| - 2(1-p)B_1|B_2| - B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2 B_1^2}{32(1+\gamma)(1+3\gamma)} \left(\frac{4p|B_3B_1 - pB_2^2| - 4(1-p)B_1[|B_2|(3-2p) + B_1] - 4B_2^2(1-p)^2 - B_1^2(1-2p)^2}{|B_3B_1 - pB_2^2| - (1-p)B_1(2|B_2| + B_1)} \right).$$

PROOF. For $f \in \mathcal{R}_\gamma^\tau(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in \mathbb{D} such that

$$(3.1) \quad 1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) = \varphi(w(z)).$$

Since f has the Maclaurin series given by (1.1), a computation shows that

$$(3.2) \quad 1 + \frac{1}{\tau}(f'(z) + \gamma z f''(z) - 1) = 1 + \frac{2a_2(1+\gamma)}{\tau}z + \frac{3a_3(1+2\gamma)}{\tau}z^2 + \frac{4a_4(1+3\gamma)}{\tau}z^3 + \dots$$

It follows from (3.1), (2.4) and (3.2) that

$$\begin{aligned} a_2 &= \frac{\tau B_1 c_1}{4(1+\gamma)}, \\ a_3 &= \frac{\tau B_1}{12(1+2\gamma)} \left[2c_2 + c_1^2 \left(\frac{B_2}{B_1} - 1 \right) \right], \\ a_4 &= \frac{\tau}{32(1+3\gamma)} [B_1(4c_3 - 4c_1c_2 + c_1^3) + 2B_2c_1(2c_2 - c_1^2) + B_3c_1^3]. \end{aligned}$$

Therefore

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{\tau^2 B_1 c_1}{128(1+\gamma)(1+3\gamma)} [B_1(4c_3 - 4c_1c_2 + c_1^3) + 2B_2c_1(2c_2 - c_1^2) + B_3c_1^3] \\ &\quad - \frac{\tau^2 B_1^2}{144(1+2\gamma)^2} \left[4c_2^2 + c_1^4 \left(\frac{B_2}{B_1} - 1 \right)^2 + 4c_2c_1^2 \left(\frac{B_2}{B_1} - 1 \right) \right] \\ &= \frac{\tau^2 B_1^2}{128(1+\gamma)(1+3\gamma)} \left\{ \left[(4c_1c_3 - 4c_1^2c_2 + c_1^4) + \frac{2B_2c_1^2}{B_1}(2c_2 - c_1^2) + \frac{B_3}{B_1}c_1^4 \right] \right. \\ &\quad \left. - \frac{8(1+\gamma)(1+3\gamma)}{9(1+2\gamma)^2} \left[4c_2^2 + c_1^4 \left(\frac{B_2}{B_1} - 1 \right)^2 + 4c_2c_1^2 \left(\frac{B_2}{B_1} - 1 \right) \right] \right\}, \end{aligned}$$

which yields

$$\begin{aligned} |a_2a_4 - a_3^2| &= T \left| 4c_1c_3 + c_1^4 \left[1 - 2\frac{B_2}{B_1} - p \left(\frac{B_2}{B_1} - 1 \right)^2 + \frac{B_3}{B_1} \right] - 4pc_2^2 \right. \\ &\quad \left. - 4c_1^2c_2 \left[1 - \frac{B_2}{B_1} + p \left(\frac{B_2}{B_1} - 1 \right) \right] \right|, \end{aligned} \quad (3.3)$$

where

$$T = \frac{|\tau|^2 B_1^2}{128(1+\gamma)(1+3\gamma)} \quad \text{and} \quad p = \frac{8(1+\gamma)(1+3\gamma)}{9(1+2\gamma)^2}.$$

It can be easily verified that $p \in [\frac{64}{81}, \frac{8}{9}]$ for $0 \leq \gamma \leq 1$.

Let

$$\begin{aligned} d_1 &= 4, \quad d_2 = -4 \left[1 - \frac{B_2}{B_1} + p \left(\frac{B_2}{B_1} - 1 \right) \right], \\ d_3 &= -4p, \quad d_4 = 1 - 2\frac{B_2}{B_1} - p \left(\frac{B_2}{B_1} - 1 \right)^2 + \frac{B_3}{B_1}. \end{aligned} \quad (3.4)$$

Then (3.3) becomes

$$|a_2a_4 - a_3^2| = T |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|. \quad (3.5)$$

It follows that

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{T}{4} |c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3) \\ &\quad + (4 - c^2)x^2(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z|. \end{aligned}$$

An application of triangle inequality, replacement of $|x|$ by μ and substituting the values of d_1 , d_2 , d_3 and d_4 from (3.4) yield

$$\begin{aligned}
 |a_2a_4 - a_3^2| &\leq \frac{T}{4} \left[4c^4 \left| \frac{B_3}{B_1} - p \frac{B_2^2}{B_1^2} \right| + 8 \left| \frac{B_2}{B_1} \right| \mu c^2 (4 - c^2)(1 - p) \right. \\
 &\quad \left. + (4 - c^2) \mu^2 (4c^2 + 4p(4 - c^2)) + 8c(4 - c^2)(1 - \mu^2) \right] \\
 (3.6) \quad &= T \left[c^4 \left| \frac{B_3}{B_1} - p \frac{B_2^2}{B_1^2} \right| + 2c(4 - c^2) + 2\mu \left| \frac{B_2}{B_1} \right| c^2 (4 - c^2)(1 - p) \right. \\
 &\quad \left. + \mu^2 (4 - c^2)(1 - p)(c - \alpha)(c - \beta) \right] \\
 &\equiv F(c, \mu)
 \end{aligned}$$

where $\alpha = 2$, $\beta = 2p/(1 - p) > 2$.

Similarly as in the previous proofs, it can be shown that $F(c, \mu)$ is an increasing function of μ for $0 < \mu < 1$. So for fixed $c \in [0, 2]$, let

$$\max F(c, \mu) = F(c, 1) \equiv G(c),$$

which is

$$\begin{aligned}
 G(c) &= T \left\{ c^4 \left[\left| \frac{B_3}{B_1} - p \frac{B_2^2}{B_1^2} \right| - (1 - p) \left(2 \left| \frac{B_2}{B_1} \right| + 1 \right) \right] \right. \\
 &\quad \left. + 4c^2 \left[2 \left| \frac{B_2}{B_1} \right| (1 - p) + 1 - 2p \right] + 16p \right\}.
 \end{aligned}$$

Let

$$\begin{aligned}
 P &= \left| \frac{B_3}{B_1} - p \frac{B_2^2}{B_1^2} \right| - (1 - p) \left(2 \left| \frac{B_2}{B_1} \right| + 1 \right), \\
 (3.7) \quad Q &= 4 \left[2 \left| \frac{B_2}{B_1} \right| (1 - p) + 1 - 2p \right], \\
 R &= 16p.
 \end{aligned}$$

Using (2.11), we have

$$|a_2a_4 - a_3^2| \leq T \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given in (3.7). □

REMARK 3.1. For the choice $\phi(z) := (1 + Az)/(1 + Bz)$ with $-1 \leq B < A \leq 1$, Theorem 3.1 reduces to [6, Theorem 2.1].

DEFINITION 3.2. Let $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ be analytic and $\varphi(z)$ as given in (2.1). For fixed real number α , function $f \in \mathcal{A}$ is in the class $\mathcal{G}_\alpha(\varphi)$ if it satisfies the following subordination:

$$(1 - \alpha)f'(z) + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \varphi(z).$$

Al-Amiri and Reade [1] introduced the class $\mathcal{G}_\alpha := \mathcal{G}_\alpha((1+z)/(1-z))$ and they have shown that $\mathcal{G}_\alpha \subset \mathcal{S}$ for $\alpha < 0$. Univalence of the functions in the class \mathcal{G}_α was also investigated in [35, 36]. Singh *et al.* also obtained the bound for the second Hankel determinant of functions in \mathcal{G}_α . The following theorem provides a bound for the second Hankel determinant of the functions in the class $\mathcal{G}_\alpha(\varphi)$.

THEOREM 3.2. *Let the function f given by (1.1) be in the class $\mathcal{G}_\alpha(\varphi)$, $0 \leq \alpha \leq 1$. Also, let*

$$p = \frac{8(1+2\alpha)}{9(1+\alpha)}.$$

(1) *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) \leq 0,$$

$$B_1^4\alpha(2\alpha-1-p\alpha) + \alpha B_1^2|B_2|(3-2p) + (\alpha+1)B_1|B_3| - p(B_1^2+B_2^2) \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9(1+\alpha)^2}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) \geq 0,$$

$$2B_1^4\alpha(2\alpha-1-p\alpha) + 2\alpha B_1^2|B_2|(3-2p) - B_1^3\alpha(3-2p)$$

$$+ 2(\alpha+1)B_1|B_3| - 2(1+\alpha-p)B_1|B_2| - (1+\alpha)B_1^2 - 2pB_2^2 \geq 0,$$

or

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) \leq 0,$$

$$B_1^4\alpha(2\alpha-1-p\alpha) + \alpha B_1^2|B_2|(3-2p) + (\alpha+1)B_1|B_3| - p(B_1^2+B_2^2) \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^4\alpha(2\alpha-1-p\alpha) + \alpha B_1^2|B_2|(3-2p) + (\alpha+1)B_1|B_3| + p(B_2^2-B_1^2)}{8(1+\alpha)(1+2\alpha)}.$$

(3) *If B_1, B_2 and B_3 satisfy the conditions*

$$B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p) > 0,$$

$$2B_1^4\alpha(2\alpha-1-p\alpha) + 2\alpha B_1^2|B_2|(3-2p) - B_1^3\alpha(3-2p)$$

$$+ 2(\alpha+1)B_1|B_3| - 2(1+\alpha-p)B_1|B_2| - (1+\alpha)B_1^2 - 2pB_2^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{32(1+\alpha)(1+2\alpha)} \left[4p - \frac{[B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p) + B_1(1+\alpha-2p)]^2}{B_1^4\alpha(2\alpha-1-p\alpha) + \alpha B_1^2|B_2|(3-2p) - B_1^3\alpha(3-2p) + (\alpha+1)B_1|B_3| - (1+\alpha-p)B_1(2|B_2|+1) - pB_2^2} \right].$$

PROOF. For $f \in \mathcal{G}_\alpha(\varphi)$, a calculation shows that

$$\begin{aligned} |a_2a_4 - a_3^2| &= T \left| 4(1+\alpha)B_1c_1c_3 + c_1^4 \left[-3\alpha B_1^2 + \alpha(2\alpha-1)B_1^3 + B_1(1+\alpha) + 3\alpha B_1B_2 \right. \right. \\ &\quad \left. \left. + (1+\alpha)(B_3-2B_2) - p \frac{(\alpha B_1^2 - B_1 + B_2)^2}{B_1} \right] - 4pB_1c_2^2 \right. \\ (3.8) \quad &\left. + 2c_1^2c_2 \left[-2(1+\alpha)B_1 + 3\alpha B_1^2 + 2(1+\alpha)B_2 - 2p(\alpha B_1^2 - B_1 + B_2) \right] \right| \end{aligned}$$

where

$$T = \frac{B_1}{128(1+\alpha)(1+2\alpha)} \quad \text{and} \quad p = \frac{8(1+2\alpha)}{9(1+\alpha)}.$$

It can be easily verified that for $0 \leq \alpha \leq 1$, $p \in [\frac{8}{9}, \frac{4}{3}]$. Let

$$\begin{aligned} (3.9) \quad d_1 &= 4(1+\alpha)B_1, \\ d_2 &= 2 \left[-2(1+\alpha)B_1 + 3\alpha B_1^2 + 2(1+\alpha)B_2 - 2p(\alpha B_1^2 - B_1 + B_2) \right], \\ d_3 &= -4pB_1, \\ d_4 &= -3\alpha B_1^2 + \alpha(2\alpha-1)B_1^3 + B_1(1+\alpha) + 3\alpha B_1B_2 + (1+\alpha)(B_3-2B_2) - p \frac{(\alpha B_1^2 - B_1 + B_2)^2}{B_1}, \end{aligned}$$

Then

$$(3.10) \quad |a_2a_4 - a_3^2| = T |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|.$$

Similar as in earlier theorems, it follows that

$$\begin{aligned} (3.11) \quad |a_2a_4 - a_3^2| &= \frac{T}{4} \left| c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4-c^2)(d_1 + d_2 + d_3) \right. \\ &\quad \left. + (4-c^2)x^2(-d_1c^2 + d_3(4-c^2)) + 2d_1c(4-c^2)(1-|x|^2)z \right| \\ &\leq T \left[c^4 \left[B_1^3\alpha(2\alpha-1-p\alpha) + \alpha B_1|B_2|(3-2p) + (\alpha+1)|B_3| - p \frac{B_2^2}{B_1} \right] \right. \\ (3.12) \quad &\left. + \mu c^2(4-c^2)[B_1^2\alpha(3-2p) + 2|B_2|(1+\alpha-p)] + 2c(4-c^2)B_1(1+\alpha) \right. \\ &\quad \left. + \mu^2(4-c^2)B_1(1+\alpha-p)(c-2) \left(c - \frac{2p}{1+\alpha-p} \right) \right] \\ &\equiv F(c, \mu), \end{aligned}$$

and for fixed $c \in [0, 2]$, $\max F(c, \mu) = F(c, 1) \equiv G(c)$ with

$$G(c) = T \left[c^4 \left[B_1^3 \alpha (2\alpha - 1 - p\alpha) + \alpha B_1 |B_2| (3 - 2p) - B_1^2 \alpha (3 - 2p) + (\alpha + 1) |B_3| \right. \right. \\ \left. \left. - (1 + \alpha - p)(2|B_2| + B_1) - p \frac{B_2^2}{B_1} \right] + 4c^2 [B_1^2 \alpha (3 - 2p) + 2|B_2|(1 + \alpha - p) \right. \\ \left. + B_1(1 + \alpha - 2p)] + 16pB_1 \right].$$

Let

$$(3.13) \quad \begin{aligned} P &= B_1^3 \alpha (2\alpha - 1 - p\alpha) + \alpha B_1 |B_2| (3 - 2p) - B_1^2 \alpha (3 - 2p) + (\alpha + 1) |B_3| \\ &\quad - (1 + \alpha - p)(2|B_2| + B_1) - p \frac{B_2^2}{B_1} \\ Q &= 4 [B_1^2 \alpha (3 - 2p) + 2|B_2|(1 + \alpha - p) + B_1(1 + \alpha - 2p)], \\ R &= 16pB_1, \end{aligned}$$

By using (2.11), we have

$$|a_2 a_4 - a_3^2| \leq T \begin{cases} R, & Q \leq 0, P \leq -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \geq 0, P \geq -\frac{Q}{8} \text{ or } Q \leq 0, P \geq -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq -\frac{Q}{8} \end{cases}$$

where P, Q, R are given in (3.13). □

REMARK 3.2. For $\alpha = 0$, Theorem 3.2 reduces to Theorem 2.2. For $0 \leq \alpha < 1$, let $\varphi(z) := (1 + (1 - 2\alpha)z)/(1 - z)$. For this function φ , $B_1 = B_2 = B_3 = 2(1 - \alpha)$. In this case, Theorem 3.2 reduces to [39, Theorem 3.1].

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